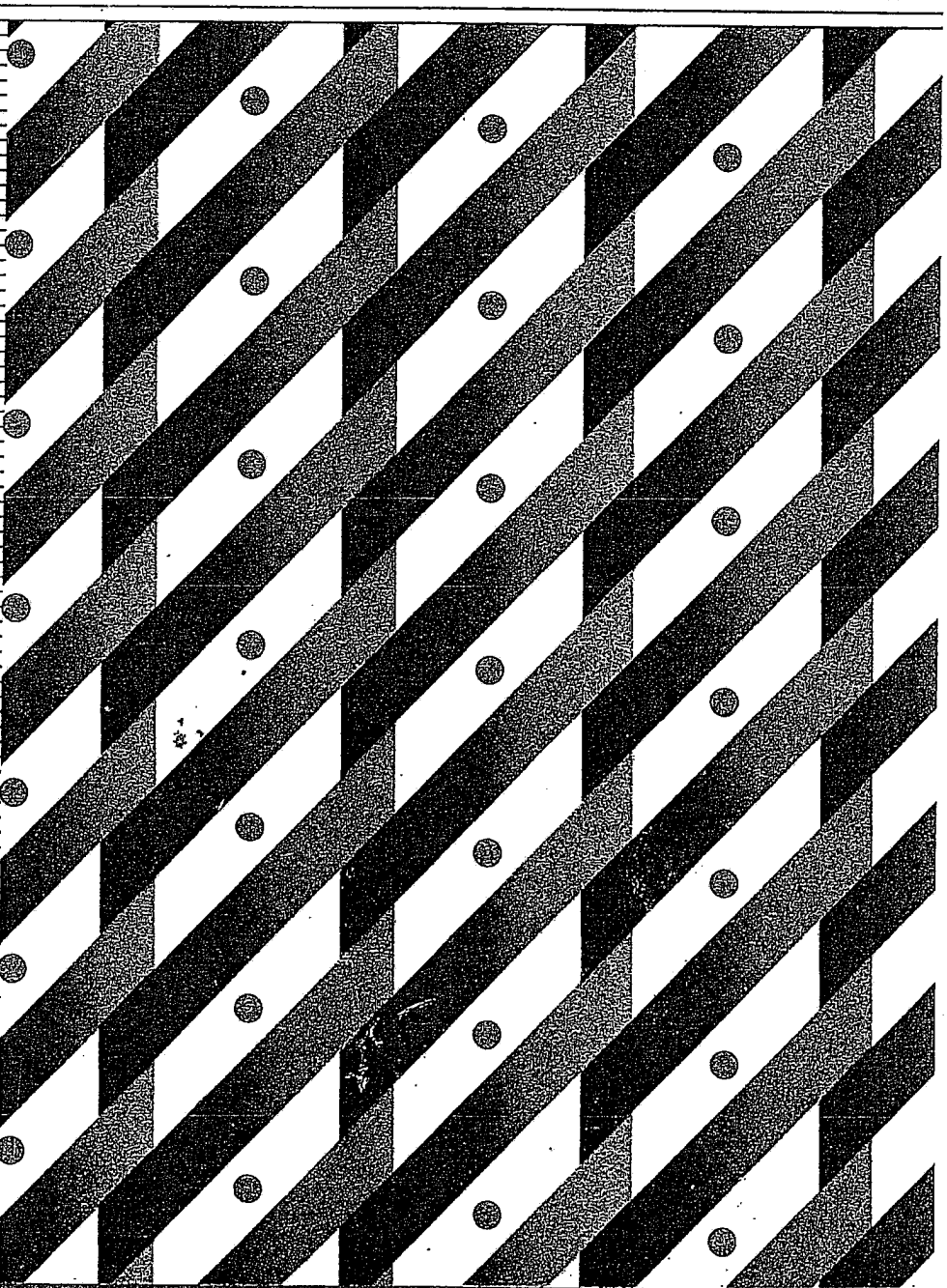
EXHIBIT 15

Probability, Random Variables, and Stochastic Processes

Athanasios Papoulis



PROBABILITY, RANDOM VARIABLES, AND STOCHASTIC PROCESSES

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CHAPTER

ONE

THE MEANING OF PROBABILITY

1-1 INTRODUCTION

The theory of probability deals with averages of mass phenomena occurring sequentially or simultaneously: electron emission, telephone calls, radar detection, quality control, system failure, games of chance, statistical mechanics, turbulence, noise, birth and death rates, and queueing theory, among many others.

It has been *observed* that in these and other fields certain averages approach a constant value as the number of observations increases and this value remains the same if the averages are evaluated over any subsequence specified before the experiment is performed. In the coin experiment, for example, the percentage of heads approaches 0.5 or some other constant, and the same average is obtained if we consider every fourth, say, tossing (no betting system can beat the roulette).

The purpose of the theory is to describe and predict such averages in terms of probabilities of events. The probability of an event $\mathscr{A}$ is a number $P(\mathscr{A})$ assigned to this event. This number could be interpreted as follows:

> If the experiment is performed n times and the event $\mathscr{A}$ occurs $n_{\mathscr{A}}$ times, then, *with a high degree of certainty*, the relative frequence $n_{\mathscr{A}}/n$ of the occurrence of $\mathscr{A}$ is *close* to $P(\mathscr{A})$:
>
> $$P(\mathscr{A}) \simeq n_{\mathscr{A}}/n \tag{1-1}$$
>
> provided that n is *sufficiently large*.

This interpretation is imprecise: the terms "with a high degree of certainty," "close," and "sufficiently large" have no clear meaning. However, this lack of

precision cannot be avoided. If we attempt to define in probabilistic terms the "high degree of certainty" we shall only postpone the inevitable conclusion that probability, like any physical theory, is related to physical phenomena only in inexact terms. Nevertheless, the theory is an exact discipline developed logically from clearly defined axioms, and when it is applied to real problems, *it works.*

Observation, deduction, prediction In the applications of probability to real problems, the following steps must be clearly distinguished:

Step 1 (physical) We determine by an inexact process the probabilities $P(\mathscr{A}_i)$ of certain events $\mathscr{A}_i$.

This process could be based on the relationship (1-1) between probability and observation: The probabilistic data $P(\mathscr{A}_i)$ equal the observed ratios $n_{\mathscr{A}_i}/n$. It could also be based on "reasoning" making use of certain symmetries: If, out of a total of N outcomes, there are $N_{\mathscr{A}}$ outcomes favorable to the event $\mathscr{A}$, then $P(\mathscr{A}) = N_{\mathscr{A}}/N$.

For example, if a loaded die is rolled 1000 times and *five* shows 203 times, then the probability of *five* equals about 0.2. If the die is fair, then, because of its symmetry, the probability of *five* equals 1/6.

Step 2 (conceptual) We assume that probabilities satisfy certain axioms, and by deductive reasoning we determine from the probabilities $P(\mathscr{A}_i)$ of certain events $\mathscr{A}_i$ the probabilities $P(\mathscr{B}_j)$ of other events $\mathscr{B}_j$.

For example, in the game with a fair die we deduce that the probability of the event *even* equals 3/6. Our reasoning is of the following form:

$$\text{If } P(1) = \cdots = P(6) = \frac{1}{6} \qquad \text{then} \qquad P(\textit{even}) = \frac{3}{6}$$

Step 3 (physical) We make a physical *prediction* based on the numbers $P(\mathscr{B}_j)$ so obtained.

This step could rely on (1-1) applied in reverse: If we perform the experiment n times and an event $\mathscr{B}$ occurs $n_{\mathscr{B}}$ times, then $n_{\mathscr{B}} \simeq nP(\mathscr{B})$.

If, for example, we roll a fair die 1000 times, our prediction is that *even* will show about 500 times.

We could not emphasize too strongly the need for separating the above three steps in the solution of a problem. We must make a clear distinction between the data that are determined empirically and the results that are deduced logically.

Steps 1 and 3 are based on *inductive reasoning.* Suppose, for example, that we wish to determine the probability of *heads* of a given coin. Should we toss the coin 100 or 1000 times? If we toss it 1000 times and the average number of heads equals 0.48 what kind of prediction can we make on the basis of this observation? Can we deduce that at the next 1000 tossings the number of heads will be about 480? Such questions can be answered only inductively.

In this book, we consider only step 2, that is, from certain probabilities we derive *deductively* other probabilities. One might argue that such derivations are mere tautologies because the results are contained in the assumptions. This is

true in the same sense that the intricate equations of motion of a satellite are included in Newton's laws.

To conclude, we repeat that the probability $P(\mathscr{A})$ of an event $\mathscr{A}$ will be interpreted as a number assigned to this event as mass is assigned to a body or resistance to a resistor. In the development of the theory, we will not be concerned about the "physical meaning" of this number. This is what is done in circuit analysis, in electromagnetic theory, in classical mechanics, or in any other scientific discipline. These theories are, of course, of no value to physics unless they help us solve real problems. We must assign specific, if only approximate, resistances to real resistors and probabilities to real events (step 1); we must also give physical meaning to all conclusions that are derived from the theory (step 3). But this link between concepts and observation must be separated from the purely logical structure of each theory (step 2).

As an illustration, we discuss in the next example the interpretation of the meaning of resistance in circuit theory.

Example 1-1 A resistor is commonly viewed as a two-terminal device whose voltage is proportional to the current

$$R = \frac{v(t)}{i(t)} \tag{1-2}$$

This, however, is only a convenient abstraction. A real resistor is a complex device with distributed inductance and capacitance having no clearly specified terminals. A relationship of the form (1-2) can, therefore, be claimed only within certain error, in certain frequency ranges, and with a variety of other qualifications. Nevertheless, in the development of circuit theory we ignore all these uncertainties. We assume that the resistance R is a precise number satisfying (1-2) and we develop a theory based on (1-2) and on Kirchhoff's laws. It would not be wise, we all agree, if at each stage of the development of the theory we were concerned with the *true* meaning of R.

1-2 THE DEFINITIONS

In this section, we discuss various definitions of probability and their role in our investigation.

Axiomatic Definition

We shall use the following concepts from set theory (for details see Chap. 2): The certain event $\mathscr{S}$ is the event that occurs in every trial. The union $\mathscr{A} + \mathscr{B}$ of two events $\mathscr{A}$ and $\mathscr{B}$ is the event that occurs when $\mathscr{A}$ or $\mathscr{B}$ or both occur. The intersection $\mathscr{A}\mathscr{B}$ of the events $\mathscr{A}$ and $\mathscr{B}$ is the event that occurs when both events $\mathscr{A}$ and $\mathscr{B}$ occur. The events $\mathscr{A}$ and $\mathscr{B}$ are mutually exclusive if the occurrence of one of them excludes the occurrence of the other.

We shall illustrate with the die experiment: The certain event is the event that occurs whenever any one of the six faces shows. The union of the events *even*